J. Fluid Mech. (1969), *vol.* 39, *part* 2, *pp.* 321–328
Printed in Great Britain

Reflexion of Kelvin waves at the open end of a rotating semi-infinite channel

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(Received 30 January 1969)

The problem of the reflexion of tides in an enclosed sea such as the North Sea at a point at which it either enters the ocean or its width suddenly increases is considered by investigating the reflexion of a Kelvin wave at the open end of a rotating uniform semi-infinite channel.

It is shown that for a given channel, if the wave period is less than a pendulum day, then, according to the linearized theory of long waves in a rotating system, the reflexion coefficient increases with the angular velocity of rotation. It is also shown that there is a resonance effect for certain critical channel widths, namely, those at which extra modes within the channel become possible.

1. Introduction

The tidal chart for the North Sea obtained by Proudman & Doodson (1924) shows very clearly that as far as the tides are concerned the North Sea may be divided into two main regions, namely the Flemish Bight and the remaining area to the north. Taylor (1920) took as a model of the second of these regions a semi-infinite canal closed at one end and considered the reflexion of a Kelvin wave at the closed end.

The situation in the Flemish Bight is more complex. First, the tide down the English coast is not totally reflected as in Taylor's model but partly transmitted around the Norfolk coast. A treatment of this problem has recently been given by Packham & Williams (1968) who have investigated the transmission of a Kelvin wave around a sharp bend in a coastline in the absence of any other boundaries. In considering the reflexion of this wave we may again use Taylor's model. This wave is, however, augmented by the tide from the English Channel along the French coast. A certain amount of this will be transmitted into the northern portion of the North Sea, but due to the sudden increase in the width of the channel some will also be reflected. One purpose of the present paper is to estimate this reflexion by taking as a model a semi-infinite open-ended channel.

The theory is, of course, equally applicable to the northern extremity of the North Sea although the effect is normally much smaller due to the relatively small amplitude of the tidal wave along the Norwegian coast. The effect could, however, be significant when the tides are augmented by a surge out of the North Sea due to a depression over the northern approaches.

The solution for the particular case of no rotation, which corresponds to that of radiation from a pair of parallel semi-infinite plates, is given in Noble (1958, p. 105).

2. Formulation of the problem

Consider a plane horizontal sheet of water of uniform undisturbed depth h rotating about a vertical axis with angular velocity $\frac{1}{2}f$. Let x and y be rectangular co-ordinates in the horizontal plane and t the time. We shall assume that the motion is governed by the linearized long wave equations

$$\frac{\partial u}{\partial t}-fv = -\frac{\partial \xi}{\partial x}, \tag{2.1}$$

$$\frac{\partial v}{\partial t}+fu = -g\frac{\partial \xi}{\partial y}, \tag{2.2}$$

$$\frac{\partial \xi}{\partial t} = -h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right), \tag{2.3}$$

where (u, v) are the velocity components in the (x, y) directions, ξ is the elevation of the surface above its mean level and g is the acceleration of gravity. We look for solutions periodic in the time t, and, assuming a time dependence $e^{-i\omega t}$, we have

$$hk^2u = -i\omega\frac{\partial \xi}{\partial x}+f\frac{\partial \xi}{\partial y}, \tag{2.4}$$

$$hk^2v = -f\frac{\partial \xi}{\partial x}-i\omega\frac{\partial \xi}{\partial y}, \tag{2.5}$$

and

$$(\nabla^2+k^2)\,\xi = 0, \tag{2.6}$$

where ∇^2 is the two-dimensional Laplacian and $k^2c^2 = \omega^2-f^2$, $c^2 = gh$.

The boundary condition at the barriers, which we take to be

$$y = \pm b, \quad -\infty < x < 0,$$

is that the normal velocity v is zero, i.e.

$$f\frac{\partial \xi}{\partial x}+i\omega\frac{\partial \xi}{\partial y} = 0 \quad \text{for} \quad y = \pm b \quad (-\infty < x < 0). \tag{2.7}$$

As incident wave we take a Kelvin wave moving from left to right in the duct given by

$$\xi = \exp\left[(i\omega x-fy)/c\right]. \tag{2.8}$$

It will be convenient to put $f = kc\sinh\beta$, $\omega = kc\cosh\beta$, and to write

$$\xi = \phi_t = \phi+\exp\left[k(ix\cosh\beta-y\sinh\beta)\right]. \tag{2.9}$$

The problem is thus to find a solution ϕ_t of (2.6) subject to the following conditions:

$$\frac{\partial \phi_t}{\partial y}-i\tanh\beta\frac{\partial \phi_t}{\partial x} = 0 \quad \text{for} \quad y = \pm b \quad (-\infty < x < 0), \tag{2.10}$$

$$\frac{\partial \phi_t}{\partial y}-i\tanh\beta\frac{\partial \phi_t}{\partial x} \quad \text{is continuous across} \quad y = \pm b \quad (-\infty < x < \infty), \tag{2.11}$$

and ϕ_t is continuous across $\quad y = \pm b, \quad x > 0. \qquad (2.12)$

3. Derivation of basic equations

Following the method and notation used by Noble (1958) for the case of no rotation we introduce the Fourier transforms

$$\Phi_t(\alpha, y) = \Phi_{t+} + \Phi_{t-} = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} \phi_t(x,y)\, e^{i\alpha x}\, dx, \quad \alpha = \sigma + i\tau,$$

$$\Phi_{t+} = \frac{1}{\sqrt{(2\pi)}}\int_{0}^{\infty} \phi_t e^{i\alpha x}\, dx, \quad \Phi_{t-} = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{0} \phi_t e^{i\alpha x}\, dx,$$

and similarly for $\phi(x,y)$.

From (2.6)
$$\frac{d^2\Phi_t}{dy^2} - \gamma^2\Phi_t = 0, \quad \gamma = (\alpha^2 - k^2)^{\frac{1}{2}},$$

and, since the incident wave is a solution of (2.6), $\Phi(\alpha, y)$ satisfies the same equation. If therefore, we set $\omega = \omega_1 + i\omega_2$, where $\omega_1 > 0$, $\omega_2 > 0$ then $k = k_1 + ik_2$, $k_2 > 0$, so that since only radiated waves exist for $|y| \geqslant b$ and ϕ consists of reflected waves which are attenuated as $x \to -\infty$, it follows that for $-k_2 < \tau < k_2$,

$$\Phi_t = A\, e^{-\gamma y} \quad (y \geqslant b), \tag{3.1}$$

$$\Phi_t = D\, e^{\gamma y} \quad (y \leqslant -b), \tag{3.2}$$

and
$$\Phi = B\, e^{-\gamma y} + C\, e^{\gamma y} \quad (|y| \leqslant b). \tag{3.3}$$

Hence for $-k_2 < \tau < k_2$,

$$\Phi_{t+}(\pm b \pm 0) + \Phi_{t-}(\pm b \pm 0) = (A, D)\, e^{-\gamma b}, \tag{3.4}$$

$$\Phi_{+}(\pm b \mp 0) + \Phi_{-}(\pm b \mp 0) = B\, e^{\mp\gamma b} + C\, e^{\pm\gamma b}, \tag{3.5}$$

$$\Phi'_{t+}(\pm b \pm 0) + \Phi'_{t-}(\pm b \pm 0) = \mp\gamma(A, D)\, e^{-\gamma b}, \tag{3.6}$$

and
$$\Phi'_{+}(\pm b \mp 0) + \Phi'_{-}(\pm b \mp 0) = -\gamma(B\, e^{\mp\gamma b} - C\, e^{\pm\gamma b}), \tag{3.7}$$

where in (A, D) the first element refers to the upper sign and the second to the lower.

Bearing in mind that for the incident wave (2.8) the normal velocity on $y = \pm b$, $-\infty < x < \infty$ is zero, it follows from (2.10) and (2.11) that

$$(A, D) = (B, C) - \frac{(\gamma \mp \alpha \tanh \beta)}{(\gamma \pm \alpha \tanh \beta)}(C, B)\, e^{2\gamma b}, \tag{3.8}$$

$$\Phi'_{t+}(\pm b \pm 0) - \alpha \tanh \beta\, \Phi_{t+}(\pm b \pm 0) = \Phi'_{+}(\pm b \mp 0) - \alpha \tanh \beta\, \Phi_{+}(\pm b \mp 0)$$
$$= \Psi_{+}(\pm b), \text{ say}, \tag{3.9}$$

$$\Phi'_{t-}(\pm b \pm 0) - \alpha \tanh \beta\, \Phi_{t-}(\pm b \pm 0) = 0 \tag{3.10}$$

and
$$\Phi'_{-}(\pm b \mp 0) - \alpha \tanh \beta\, \Phi_{-}(\pm b \mp 0) = 0. \tag{3.11}$$

Similarly from (2.12)

$$\Phi_{t+}(\pm b \pm 0) = \Phi_{+}(\pm b \mp 0) + \frac{i \exp[\mp kb \sinh \beta]}{\sqrt{(2\pi)}\,(\alpha + k\cosh\beta)}. \tag{3.12}$$

From (3.4), (3.6), (3.9) and (3.10) (or alternatively (3.5), (3.7), (3.9) and (3.11)) it then easily follows that

$$\Psi_{+}(\pm b) = \mp(\gamma \pm \alpha \tanh \beta)(A, D)\, e^{-\gamma b}$$
$$= (\gamma - \alpha \tanh \beta)\, C\, e^{\pm\gamma b} - (\gamma + \alpha \tanh \beta)\, B\, e^{\mp\gamma b}. \tag{3.13}$$

If now we write

$$2F_-(\pm b) = \Phi_{t-}(\pm b \pm 0) - \Phi_-(\pm b \mp 0), \tag{3.14}$$

$$(S'_+, D'_+) = \Psi_+(b) \pm \Psi_+(-b), \tag{3.15}$$

$$(S_-, D_-) = F_-(b) \pm F(-b), \tag{3.16}$$

then from (3.4), (3.5) and (3.12)

$$F_-(\pm b) + \frac{i \exp[\pm kb \sinh\beta]}{2\sqrt{(2\pi)}\,(\alpha + k\cosh\beta)} = -(C, B)\frac{\gamma\, e^{\gamma b}}{\gamma \pm \alpha\tanh\beta}, \tag{3.17}$$

so that

$$(D_-, S_-) = \pm\frac{\gamma\, e^{\gamma b}}{(\gamma^2 - \alpha^2\tanh^2\beta)}\{(\gamma + \alpha\tanh\beta)B \mp (\gamma - \alpha\tanh\beta)C\}$$
$$\pm \frac{i}{\sqrt{(2\pi)}\,(\alpha + k\cosh\beta)}\frac{\sinh}{\cosh}(kb\sinh\beta), \tag{3.18}$$

and from (3.13)

$$(S'_+, D'_+) = (e^{\gamma b} \pm e^{-\gamma b})\{(\gamma - \alpha\tanh\beta)C \mp (\gamma + \alpha\tanh\beta)B\}. \tag{3.19}$$

Hence

$$(S'_+, D'_+) = -\frac{(\gamma^2 - \alpha^2\tanh^2\beta)}{\gamma}(1 \pm e^{-2\gamma b})$$
$$\times\left\{(D_-, S_-) \mp \frac{i}{\sqrt{(2\pi)}\,(\alpha + k\cosh\beta)}\frac{\sinh}{\cosh}(kb\sinh\beta)\right\},$$

or since $\gamma^2 - \alpha^2\tanh^2\beta = (\alpha^2 - k^2\cosh^2\beta)/\cosh^2\beta$,

$$(S'_+, D'_+) = -\frac{(\alpha^2 - k^2\cosh^2\beta)}{\cosh^2\beta}\left(\frac{1 \pm e^{-2\gamma b}}{\gamma}\right)$$
$$\times\left\{(D_-, S_-) \mp \frac{i}{\sqrt{(2\pi)}\,(\alpha + k\cosh\beta)}\frac{\sinh}{\cosh}(kb\sinh\beta)\right\}. \tag{3.20}$$

These two equations are of Wiener–Hopf type and are soluble for (S'_+, D'_+) in the normal way using known factorizations (cf. Heins 1948 and Noble 1958).

4. Solution of basic equations

Employing Noble's notation we suppose that we may write

$$\frac{1 - e^{-2\gamma b}}{2b\gamma} = L(\alpha) = L_+(\alpha)\,L_-(\alpha), \tag{4.1}$$

and

$$\frac{1 + e^{-2\gamma b}}{2} = e^{-\gamma b}\cosh\gamma b = K(\alpha) = K_+(\alpha)\,K_-(\alpha), \tag{4.2}$$

where K_+, L_+ are regular in $\tau > -k_2$, K_-, L_- are regular in $\tau < k_2$, and $|K_-|$, $|K_+|$ are asymptotic to constants, and $|L_-|$, $|L_+|$ are asymptotic to $|\alpha|^{-\frac{1}{2}}$ in appropriate half-planes. Explicit factorizations and deductions from them are given in some detail by Noble (1958) and we will therefore merely quote the appropriate formulae as they are required. In particular we note that

$$K_+(-\alpha) = K_-(\alpha), \tag{4.3}$$

and

$$L_+(-\alpha) = L_-(\alpha). \tag{4.4}$$

Taking the upper sign in (3.20) the equation may be written in the form

$$\frac{S'_+(\alpha+k)^{\frac{1}{2}}}{(\alpha+k\cosh\beta)K_+(\alpha)}-\frac{2\sqrt{(2k)}\sinh(kb\sinh\beta)K_-(-k\cosh\beta)}{\sqrt{(2\pi)}\cosh\beta\cosh\frac{1}{2}\beta(\alpha+k\cosh\beta)}$$
$$=\frac{-2(\alpha-k\cosh\beta)K_-(\alpha)D_-}{(\alpha-k)^{\frac{1}{2}}\cosh^2\beta}+\frac{2i\sinh(kb\sinh\beta)}{\sqrt{(2\pi)}\cosh^2\beta(\alpha+k\cosh\beta)}$$
$$\times\left\{(\alpha-k\cosh\beta)\frac{K_-(\alpha)}{(\alpha-k)^{\frac{1}{2}}}+\frac{i\sqrt{(2k)}\cosh\beta K_-(-k\cosh\beta)}{\cosh\frac{1}{2}\beta}\right\}. \quad (4.5)$$

The left-hand side is regular in $\tau > -k^2$ and the right-hand side is regular in $\tau < k_2$, and each term tends to zero as $\alpha\to\infty$ in the appropriate half-plane. Hence, by the usual arguments based on Liouville's theorem, each side of the equation is zero and

$$S'_+ = \frac{2\sqrt{(2k)}\sinh(kb\sinh\beta)K_-(-k\cosh\beta)}{\sqrt{(2\pi)}\cosh\beta\cosh\frac{1}{2}\beta(\alpha+k)^{\frac{1}{2}}}K_+(\alpha),$$

or, using (4.3),

$$S'_+ = \frac{2\sqrt{(2k)}\sinh(kb\sinh\beta)K_+(k\cosh\beta)}{\sqrt{(2\pi)}\cosh\beta\cosh\frac{1}{2}\beta}\frac{K_+(\alpha)}{(\alpha+k)^{\frac{1}{2}}}. \quad (4.6)$$

Similarly, taking the lower sign in (3.20) the equation may be written in the form

$$\frac{D'_+}{(\alpha+k\cosh\beta)L_+(\alpha)}-\frac{i4bk\cosh(kb\sinh\beta)L_-(-k\cosh\beta)}{\sqrt{(2\pi)}\cosh\beta(\alpha+k\cosh\beta)}$$
$$=-\frac{2b(\alpha-k\cosh\beta)}{\cosh^2\beta}L_-(\alpha)S_--\frac{i2b\cosh(kb\sinh\beta)}{\sqrt{(2\pi)}\cosh^2\beta(\alpha+k\cosh\beta)}$$
$$\times\{(\alpha-k\cosh\beta)L_-(\alpha)+2k\cosh\beta L_-(-k\cosh\beta)\}. \quad (4.7)$$

Again, by the usual arguments, each side of the equation is zero, and therefore

$$D'_+ = \frac{i4bk\cosh(kb\sinh\beta)L_-(-k\cosh\beta)}{\sqrt{(2\pi)}\cosh\beta}L_+(\alpha),$$

or using (4.4)

$$D'_+ = \frac{i4bk\cosh(kb\sinh\beta)L_+(k\cosh\beta)}{\sqrt{(2\pi)}\cosh\beta}L_+(\alpha). \quad (4.8)$$

5. The reflected wave

In order to determine the reflected wave we require ϕ in the region

$$|y| \leqslant b, \quad x < 0.$$

Now for $|y| \leqslant b$, we have from (3.3) that

$$\Phi(\alpha, y) = B\,e^{-\gamma y}+C\,e^{\gamma y},$$

where from (3.13) and (3.15), B and C are given by

$$(\gamma-\alpha\tanh\beta)\,C\,e^{\pm\gamma b}-(\gamma+\alpha\tanh\beta)\,B\,e^{\pm\gamma b} = \tfrac{1}{2}(S'_+\pm D'_+),$$

or

$$(B, C) = \frac{1}{4(\gamma\pm\alpha\tanh\beta)}\left\{\frac{D'_+}{\sinh\gamma b}\mp\frac{S'_+}{\cosh\gamma b}\right\}.$$

Hence

$$\Phi(\alpha, y) = \frac{1}{4(\gamma^2 - \alpha^2 \tanh^2 \beta)} \left\{ (\gamma - \alpha \tanh \beta) \left[\frac{D'_+}{\sinh \gamma b} - \frac{S'_+}{\cosh \gamma b} \right] e^{-\gamma y} + (\gamma + \alpha \tanh \beta) \left[\frac{D'_+}{\sinh \gamma b} + \frac{S'_+}{\cosh \gamma b} \right] e^{\gamma y} \right\}. \quad (5.1)$$

Taking the Fourier inverse

$$\phi(x, y) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty + i\tau}^{\infty + i\tau} \Phi(\alpha, y)\, e^{-i\alpha x}\, d\alpha. \quad (5.2)$$

In the region $x < 0$ we can close the contour in the upper half-plane. The singularities of $\Phi(\alpha, y)$ in the upper half-plane are a simple pole at $\alpha = k \cosh \beta$, which gives a reflected Kelvin wave, and the zeros of $\cosh \gamma b$ and $\sinh \gamma b/(\gamma b)$. If we let $k_2 \to 0$ then for the case $0 < 2kb < \pi$ all the zeros of $\cosh y\gamma$ and $\sinh \gamma b/(\gamma b)$ are imaginary and give rise to waves which are propagated to the left and tend to zero exponentially as $x \to -\infty$. We will restrict our investigation to the calculation of the amplitude of the reflected Kelvin wave for this case which includes the most interesting physical problems.

Now it is easily seen from (5.1) that the contribution to (5.2) from the pole at $\alpha = k \cosh \beta (\gamma = k \sinh \beta)$ is a Kelvin wave moving from right to left in the duct given by

$$A \exp[ky \sinh \beta - ikx \cosh \beta] = A \exp[-(i\omega x - fy)/c], \quad (5.3)$$

where, from (5.1), (4.6) and (4.8)

$$A = \frac{i \sinh \beta}{2} \left[\frac{2i\, bk \cosh (kb \sinh \beta)\, L^2_+(k \cosh \beta)}{\sinh (kb \sinh \beta)} + \frac{\sinh (kb \sinh \beta)\, K^2_+(k \cosh \beta)}{\cosh^2 \frac{1}{2}\beta \cosh (kb \sinh \beta)} \right]. \quad (5.4)$$

If now we put $b_1 = kb/\pi$ and use the formulae given in Noble (1958) for $L^2_+(k \cosh \beta)$ and $K^2_+(k \cosh \beta)$ it follows from (2.8) and (5.3), after some manipulation, that for $0 < b_1 < \frac{1}{2}$ the reflexion coefficient R (the ratio of the amplitude of the reflected wave to that of the incident wave) is given by

$$R = |A| = \cosh (\pi b_1 \sinh \beta) \exp\{-\pi b_1 \cosh \beta\} \times |1 - i \tanh \tfrac{1}{2}\beta \tanh (\pi b_1 \sinh \beta)\, e^{2im}|,$$

where

$$m = \sum_{n=1}^{\infty} \{\Psi_n - \Psi_{n-\frac{1}{2}}\},$$

and

$$\tan \Psi_n = \frac{b_1 \cosh \beta}{(n^2 - b_1^2)^{\frac{1}{2}}}.$$

We observe that in the absence of rotation ($\beta = 0$) this reduces to

$$R = \exp(-\pi b_1),$$

which agrees with Noble's result.

In order to show the dependence of R on the width of the channel we have plotted R against b_1 ($0 < b_1 < \frac{1}{2}$) for values of β corresponding to $f/\omega = 0, \frac{2}{5}, \frac{2}{3}, \frac{1}{2}\sqrt{3}$, 0·925, 0·99 and 0·999.

We note that the upturn in the graphs as b_1 approaches $\frac{1}{2}$ is associated with the appearance of a second unattenuated reflected mode inside the channel as b_1 passes through this critical value and the graphs for the dominant mode in fact have a cusp at this point.

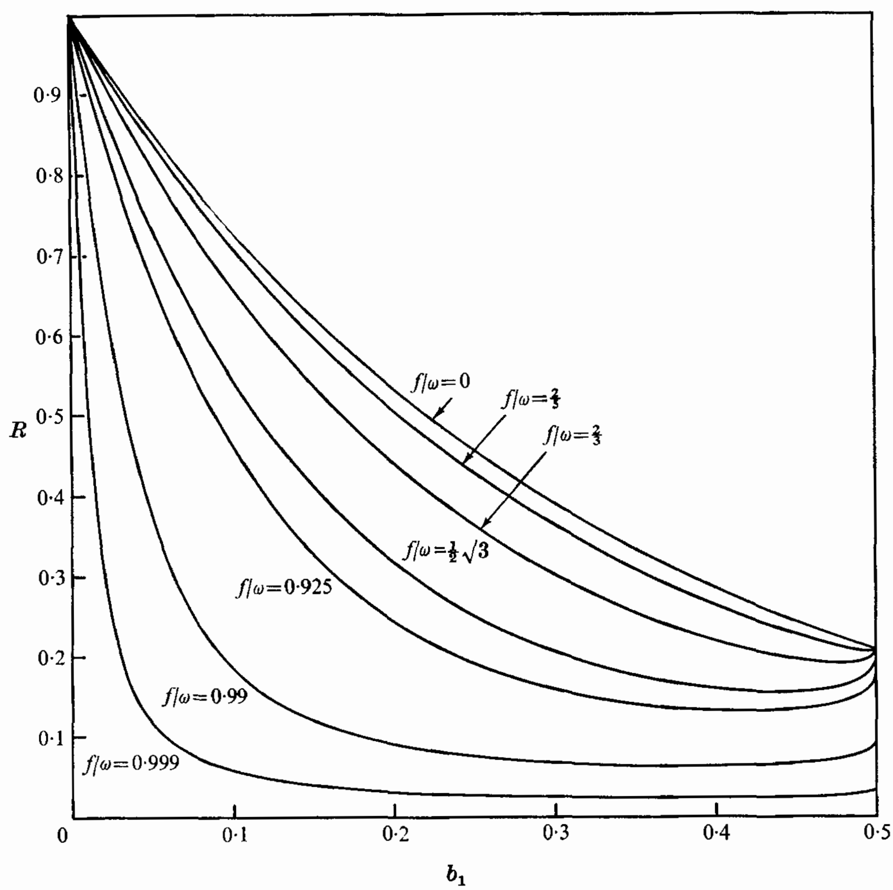

FIGURE 1. The reflexion coefficient R as a function of b_1 for $f/\omega = 0, \frac{2}{5}, \frac{2}{3}, \frac{1}{2}\sqrt{3}$, 0·925, 0·99 and 0·999.

We also note that for a given wave frequency and depth of water b is proportional to $b_1 \cosh\beta$, so that, although for fixed b_1 the reflexion coefficient R decreases as β increases, for a given channel, R increases with increasing angular velocity.

The value $f/\omega = \frac{1}{2}\sqrt{3}$ corresponds approximately to the semi-diurnal tide M_2 at the entrance to the North Sea. With this value of f/ω the appropriate value of b_1 for the northern portion of the North Sea (with $b = 200\,\text{km}$, $h = 100\,\text{m}$) is approximately 0·145, and the corresponding value of R is approximately 0·42. Since the amplitude of the tide on the Norwegian coast is of the order of 30 cm the theory predicts a reflected wave whose amplitude is of the order of 12 cm. Considerably large amplitudes may, however, be obtained when the tide is augmented by a surge out of the North Sea due to a depression over the northern approaches and the reflected wave may then be significant.

For the Flemish Bight the appropriate value of b_1 (with $b = 50\,\text{km}$, $h = 30\,\text{m}$) is approximately 0·07 and the corresponding value of R is approximately 0·58. Since the amplitude of the tide on the Dutch coast is of the order of 80 cm we expect a reflected wave of the order of 46 cm down the Norfolk coast. This is about half the observed tide and the observed values are obtained if about 30 % of the East coast tide is transmitted around the Norfolk coast and the remainder is transmitted directly to the Flemish coast as in Taylor's (1920) model.

I am indebted to Mr T. Fuller for his invaluable help with the numerical computations.

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